New Type of Open Sets in Bitopological spaces

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Abstract:

Many types of open set in Bitopological space defined by the mathematician in our paper new type of open set in Bitopological spaces defined in which is called K-open set.

1. Introduction:

A Bitopological Spaces (X , T_1, T_2) [1] is anon empty set X with two topologies T_1 and T_2 on X. Bitopological spaces defined by J. N. Kelly [2]. After that a different definition and studies in Bitopological spaces appears.

In this research we define a new set in Bitopological Space which called it K-open set and by using this set we define K-connected, K-compact, K-continuous and K-separation axiom. We study some theorems by using these definitions.

2. Basic Definition:

Definition 2.1: Let X be any non –empty set and T_1, T_2 are two topological space on X, then (X, T_1, T2) be a Bitopological space. A sub set A of X is called to be K- open set in Bitopological space X if for each $x \in$ A $\exists$ u $\in$ T_1 such that int T_2(u) $\subseteq$ A.

Example 2.1: Let X = {a, b, c} , T_1 = { X, ϕ , {a}, {b}, {a, b}} and T_2 = { X, ϕ , {a}, {c}, {a, c}} then K.O(X) = {X, ϕ , {a}, {a, b}, {a, c}, {b, c}}.

Example 2.2: Let X = {a, b, c} , T_1 = { X, ϕ, {a}, {b}, {a, b}} and T_2 = { X, ϕ , {a, c}} then K.O(X) = {X, ϕ }.

Remark 2.1: The set of all open sets is a topological space.

Remark 2.2:

a- If A is T_i – open set , i = 1, 2 then A is K- open set.

b- If A is T_i – open set , i = 1 or 2 then it is not necessary that A is K- open set.

Example 2.4: See example (2.1) and (2.2).

Remark 2.3: If $T_1 \subset T_2$ then every T_1– open set is open set in Bitopological space (X, T_1,T_2).

Theorem 2.1: Every T_1 - open set is open set in Bitopological space (X, T_1,T_2).

Proof: Let A $\subseteq$ X and A is T_1 – open set then int T_2(A) $\subset$ A Then A is K – open set.

3. Continuous:

Definition 3.1: A mapping F: (X, T_1, T2) $\rightarrow$ (Y , P_1,P_2) is said to be

1. K – continuous if for each $x \in$ X and K – open set V in Y such that F(x) $\in V$ there exist K – open set U such that F(U) $\in V$.
2. K- open mapping iff F(U) is K – open set in Y for each K – open set U in X.
3. K- closed mapping iff F(U) is K – closed set in Y for each K – closed set U in X.
4. K – Homeomorphism iff F: $x \rightarrow y$ is K- continuous, 1-1.

Theorem 3.1: If g: $(X , T_1, T_2) \rightarrow (Y , P_1, P_2)$ is K- continuous then g: $(X , T_1, T_2) \rightarrow (Y , P_1)$ is continuous.

Proof: Let U is P_1– open set then U is K-open set in Y and since g is K-continuous. Then $g^{-1}(U)$ is K– open set in X.

Theorem 3.2: the composition of two K-continuous functions is K-continuous.

Proof: The same proof in Sharma with replace open set by K-open set.

4. Connected and Compact:

Definition 4.1: A sub sets A, B of X are said to be separated in (X , T_1,T_2) iff A, B are T_i– separated, i = 1, 2.

Example 4.1: Let X = { a, b, c}, T_1 = { X, ϕ, {a}, {b, c}}= T_2 , {a}, {b, c} are separated in (X , T_1,T_2)

Definition 4.2: A Bitopological space (X , T_1,T_2) is say to be connected iff for each two K.O(X) A and B then $A \cap B = \phi$, $A \cap CL(B) = \phi$ and $B \cap CL(A) = \phi$.

Example 4.2 : Let X = {a, b, c} , T_1 = { X, ϕ, {a}} and T_2 = { X, ϕ, {c}, {a, b}} are connected topologies space. Then K.O(X) = {X, ϕ, {a}, {a, b}, {a, c}}.

Remark 4.1 : If (X, T_i) is a connected topological spaces, i = 1, 2 then it is not necessary that (X , T_1,T_2) is a connected topological space.

Example 4.3 : See example (2.2).

Definition 4.3: A Bitopological space (X , T_1, T_2) is say to be compact iff (X, T_i) is compact, i = 1, 2.

Example 4.4 : Let X = {a, b}, T_1 = { X, ϕ }, T_2 = { X, ϕ, {a}}. Then (X , T_1,T_2) is a compact Bitopological space.

5. Separation Axiom:

Definition 5.1 : A Bitopological space (X , T_1, T_2) is said to be K- T_0 (re sp K- T_1, K- T_2) iff for each two points *x*, *y* in X there exist K- T_0– open set *G* such that $x \in G$, $y \notin G$ or there exist K- T_0 – open set *H* such that $x \notin H$, $y \in H$. (rep for each two points *x*, *y* in X, there exist two K- open sets *G, H* such that $x \in G$, $y \notin G$, $x \notin H$, $y \in H$, for each two points *x*, *y* in X there exist two K- open sets *G, H* such that $x \in G$, $y \in H$, $G \cap H = \phi$).

Example 5.1 : Let (X, P_1, P_2) be a Bitopological space and If P_2 is finer than P_1 and P_2 is T_0 (resp T_1, T_2) then (X, T_1, T_2) is K- T_0 (re sp K- T_1, K- T_2-space)

Example 5.2: Let X = {a, b, c} , P_1 = { X, ϕ, {a}} and P_2 = { X, ϕ, {a},{b}, {a, b}, {c}, {a, c}, {b, c}}. Then (X , T_1,T_2) is K- T_0, K- T_1 and K- T_2, where P_2 is T_0, T_1 and T_2.

Definition 5.2: A Bitopological space (X , T_1, T_2) is said to be K- regular space iff for each $x \in X$ and T_i – closed set F, i = 1 or 2 such that $x \notin F$. There exist two K-open sets G, H such that $x \in G$, $F \subset H$, $G \cap H = \phi$.

Example 5.3: If P_2 is finer than P_1 then if P_2 regular space .Then (X , T_1, T_2) is K-regular space.

Definition 5.3: A Bitopological space (X , P_1, P_2) is said to be K- T_3 space if (X , P_1, P_2) is K- T_1 and K- regular space.

Theorem 5.1: Let (X , P_1, P_2) be K- T_1 Bitopological space $Y \subseteq X$ is K-open set of X then (Y , T_{1Y}, T_{2Y}) is K-T_1 space.

Proof: Let $(Y , P_{1Y,} P_{2Y})$ is subspace of $(X , P_{1,} P_2)$ and $y_1, y_2 \in Y$ then $y_1, y_2 \in X$, X is K- $T_1 \Rightarrow \exists$ K-open set U such that $y_1 \in U, y_2 \notin U \Rightarrow y_1 \in U \cap Y, y_2 \notin U \cap Y$ and since $U \cap Y$ is K-open set $\Rightarrow (Y , T_{1Y,} T_{2Y})$ is K- T_1 space.

Theorem 5.2: Let $(X , P_{1,} P_2)$ be K- regular Bitopological space and $Y \subseteq X$ such that Y is K-open set .then $(Y , T_{1Y,} T_{2Y})$ is regular space.

Proof: Let $y \in Y$, F is T_i – closed set, i =1 or 2 such that $y \notin F$ then since $y \in X$ and F is T_i – closed set. Then $\exists$ U_1, U_2 are K-open sets such that $y \in U_1$, $F \subseteq U_2$, $U_1 \cap U_2 = \phi$. Thus $(Y , T_{1Y,} T_{2Y})$ is K- regular space.

Theorem 5.3: If (X , P_1) is T_0, T_1, T_2 space then $(X , P_{1,} P_2)$ is K- T_0 (K-$T_{1,}$ K-T_2).

Proof: Let $x, y \in X$ and $x \neq y$. Since (X , P_1) is T_0(resp. $T_{1,}$ T_2) then $\exists$ P_1-open set u (resp. $\exists$ P_1-open set $u_{1,}$ $u_{2,}$ $\exists$ P_2-open set $u_{1,}$ u_2) such that $x \in u$, $y \notin u$(resp. $x \in u_{1,}$ $y \in u_{2,}$ $y \notin u_{1,}$ $x \notin u_2$, $x \in u_1$, $y \in u_{2,}$ $u_1 \cap u_2 = \phi$ **)** and since every P_1-open set is K- open set then $(X , P_{1,} P_2)$ is K- T_0 (K-$T_{1,}$ K-T_2).

Definition 5.4: A Bitopological space $(X , T_{1,} T_2)$ is said to be Normal space iff for each two T_i – closed sets F_1, F_2, i= 1 or 2. There exist two K- open sets G, H such that $F_1 \subseteq G$, $F_2 \subseteq H$, $G \cap H = \phi$.

Theorem 5.4: If (X, T_1) is regular (resp. Normal) then $(X , T_{1,} T_2)$ is regular (resp. Normal).

Proof: Let $x \in X$ and F is T_1- closed set such that $x \notin F$. Since (X , T_1) is regular then there exist two open sets G, H such that $x \in G$, $F \subseteq H$, $G \cap H = \phi$. And since every T_1 – open set is K- open set then $(X, T_{1,} T_2)$ is K- regular space. Now to prove the normality, let F_1, F_2 are T_1 – closed sets. Since (X, T_1) is Normal space, there exist two T_1 – open sets G, H such that $F_1 \subseteq G$, $F_2 \subseteq H$, and $G \cap H = \phi$. Since every T_1 – open set is K- open set then $(X, T_{1,} T_2)$ is K- Normal space.

Theorem 5.5: K- T_0, K-T_1, K-T_2, K-T_3, K-T_4 are topological properties.

Proof: We use the same proof in [3] with replacing every open set by K – open set.

Theorem 5.6: Let $(X , T_{1,} T_2)$ be a Bitopological space then $(X , T_{1,} T_2)$ is K- T_1- space iff every singleton subset of X is closed set.

Proof: The same proof in Sharma with replace every open set by K-open set.

Definition 5.5: Let $(X , T_{1,} T_2)$ be a Bitopological space and $Y \subseteq X$ then the subspace defined by $(Y , T_{1Y,} T_{2Y})$ and $T_{1Y} = \{G \cap Y : G \text{ is } T_1 \text{ - open set}\}$, $T_{2Y} = \{H \cap Y : H \text{ is } T_2 \text{ - open set}\}$.

Theorem 5.7: A T_i – closed subspace $(Y , T_{1Y,} T_{2Y})$ of a normal Bitopological space $(X , T_{1,} T_2)$ is normal when Y is K-open set.

Proof: Let Y is T_i – closed set and B_1, B_2 are T_{iY} – closed set, i = 1 or 2. Since $B_1 = Y \cap F_{1,}$ $B_2 = Y \cap F_{2,}$ Y is T_i – closed set and $(X , T_{1,} T_2)$ is normal, $\Rightarrow \exists$ K-open sets H_1, H_2 such that $B_1 \subseteq U$, $B_2 \subseteq V_,$ $H_1 \cap H_2 = \phi_.$ $Y \cap F_1 \subseteq H_1$, $Y \cap F_2 \subseteq H_2$ $\Rightarrow$ $B_1 \subseteq U \cap Y$, $B_2 \subseteq V \cap Y$.

Reference:

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[3] J. N. Sharama , " Topology " , Meerut College, Meerut, 1977.