Solving (O.D.E)By New Transformation

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الملخص

الهدف من هذا البحث هو أيجاد تحويل تكاملي جديد و تطبيقات على هذا التحويل وذلك باستخدامه في أيجاد الحل العام للمعادلة التفاضلية الاعتيادية الخطية ذات المعاملات الثابتة من الرتبة (m) بدون استخدام الشروط الابتدائية.

ABSTRACT

The aim of this paper is to find new integral transformation and applications on it, by using this new transformation in finding the general solution of linear ordinary differential equations (L.O.D.Es)with constant coefficients of order (m) ,without using any initial conditions(I.C) .

1.INTRODUCTION

We can solve the ordinary differential equations by using integral transformations, (Laplace transformation) by using initial and boundary condition .
Also, we can find the general solution of linear ordinary differential equations without using any initial conditions and by using Laplace transformations.

In this paper, we search for a new integral transformation by finding a new transformation kernel $[k(x,s)=a^{-sx}$ when a positive integer $, a \neq 1 , 2.7]$ for $(L^{*}T)$ and it can be used it to solve the linear ordinary differential equations with constant coefficients of order (m) without using initial and boundary conditions and by taking $(L^{*}T)$ of both sides and substituting $(L^{*}T)$ on the ordinary derivatives, and taking inverse $(L^{*}T)$ .So we can obtain the solution of linear ordinary differential equations with constant coefficients of order (m) without using any initial and boundary conditions.

2.Definitions

***Definition(2.1)*:-**

An ordinary differential equation is a relation among an independents variable x ,an unknown function $y(x)$ of that variable, and certain of its derivatives. There for, the most general form in which an ordinary differential equation may assume is :

$$G\left(x , y , y', \ldots y^{(n)}\right)=0 \quad \ldots (1)$$

***Definition(2.2)*:-**

To obtains the $(L^{*}T)$ of a function $g(x)$,we multiply the function $g(x)$ by a^{-sx} $\ni a \neq 1,2.7$ and integrate the product ($a^{-sx} g(x)$) with respect to the (x) between $x = 0$ and $x = \infty$,i.e.

$$L^{*}(g(x)) = \int_{0}^{\infty} a^{-sx} g(x) dx \qquad \dots (2)$$

The constant parameter, s , is assumed to be positive and larger enough to make the product $a^{-sx} g(x)$ converges to zero as $x \to \infty$,
not that (2) is a definition integral with limits to be substituted for x so that the resulting expression will not contain x but will be expressed in terms of, s , only i.e.

$$L^{*}(g(x)) = \int_{0}^{\infty} a^{-sx} g(x) dx = \overline{G}(s)$$

Definition(2.3):-

Let $g(x)$ be a function of (x) ,and $L^{*}(g(x)) = \overline{G}(s)$, $g(x)$ is said to be an inverse for the $(L^{*}T)$ and it is written as:

$$g(x) = \left(L^{*}\right)^{-1}(\overline{G}(s))$$

where $\left(L^{*}\right)^{-1}$ returns the transformation to the original function.

3. PROPERTIES

Property(3.1):-

If $g_1(x), g_2(x), \dots, g_n(x)$ are functions such that $x > 0$ and if $c_1, c_2, \dots, c_n$ are constants then:

$$L^{*}[c_1 g_1(x) + c_2 g_2(x) + \dots + c_n g_n(x)] = c_1 L^{*}[g_1(x)] + c_2 L^{*}[g_2(x)] + \dots + c_n L^{*}[g_n(x)]$$

Property(3.2):-

If $y^{(m)}(x)$ represents a derivative of $y(x)$ with respect to (x) of order (m) then:-

$$L^{*}[y^{m}] = (s \ln a)^{m} L^{*}(y) - (s \ln a)^{m-1} y(0) - (s \ln a)^{m-2} y'(0) - \dots - y^{(m-1)}(0)$$

$\ni a \neq 1,2.7$

4. Transformations for { $g(x)$ }

ID	Some function of ($g(x)$)	$L^{*}(g(x)) = \overline{G}(s)$, $a \neq 1,2.7$

1	C , c =constant	$L^{*}(c)=\frac{c}{s\ln a}$
2	e^{bt}	$L^{*}(e^{bt})=\frac{1}{s\ln a-b}$
3	a^{bt}	$L^{*}(a^{bt})=\frac{1}{(s-b)\ln a}$
4	t^{n} , $n\in z^{+}$	$L^{*}(t^{n})=n!\,\frac{1}{(s\ln a)^{n+1}}$
5	$\sin bt$	$L^{*}(\sin bt)=\frac{b}{(s\ln a)^{2}+b^{2}}$
6	$\cos bt$	$L^{*}(\cos bt)=\frac{s\ln a}{(s\ln a)^{2}+b^{2}}$
7	$\sinh bt$	$L^{*}(\sinh bt)=\frac{b}{(s\ln a)^{2}-b^{2}}$
8	$\cosh bt$	$L^{*}(\cosh bt)=\frac{s\ln a}{(s\ln a)^{2}-b^{2}}$

Proof:-

1) $L^{*}(c)=\int_{0}^{\infty} c\,a^{-st}\,dt \Rightarrow =c\int_{0}^{\infty} a^{-st}\,dt \Rightarrow =\frac{-c}{s\ln a}a^{-st}\Big|_{0}^{\infty}$

$L^{*}(c)=\frac{c}{s\ln a}$

2) $L^{*}(e^{bt})=\int_{0}^{\infty} e^{bt}a^{-st}\,dt \Rightarrow =\int_{0}^{\infty} e^{bt}e^{-st\ln a}\,dt \Rightarrow =\int_{0}^{\infty} e^{-(s\ln a-b)t}\,dt \Rightarrow =\frac{-1}{s\ln a-b}e^{-(s\ln a-b)t}\Big|_{0}^{\infty}$

$L^{*}(e^{bt})=\frac{1}{s\ln a-b}$

3) $L^{*}(a^{bt})=\int_{0}^{\infty} a^{bt}a^{-st}\,dt \Rightarrow =\int_{0}^{\infty} a^{-(s-b)t}\,dt \Rightarrow =\frac{-1}{(s-b)\ln a}a^{-(s-b)t}\Big|_{0}^{\infty}$

$L^{*}(a^{bt})=\frac{1}{(s-b)\ln a}$

4) $L^{*}(t^{n})=\int_{0}^{\infty} t^{n}a^{-st}\,dt$ $\Rightarrow$ by using the same above method $\Rightarrow$ $=n!\,\frac{-1}{(s\ln a)^{n+1}}a^{-st}\Big|_{0}^{\infty}$

$L^{*}(t^{n})=n!\,\frac{1}{(s\ln a)^{n+1}}$

5) $L^{*}(\sin bt)=\int_{0}^{\infty} \sin bt\,a^{-st}\,dt \Rightarrow =\int_{0}^{\infty} \frac{e^{ibt}-e^{-ibt}}{2i}a^{-st}\,dt$ $\Rightarrow$ by using (2) we get

$$L^*(\sin bt)=\frac{b}{(s\ln a)^2+b^2}$$

6) $L^*(\cos bt)=\int_0^\infty \cos bt\, a^{-st}dt \Rightarrow =\int_0^\infty \frac{e^{ibt}+e^{-ibt}}{2}a^{-st}dt \Rightarrow$ by using (2) we get

$$L^*(\cos bt)=\frac{s\ln a}{(s\ln a)^2+b^2}$$

7) $L^*(\sinh bt)=\int_0^\infty \sinh bt\, a^{-st}dt \Rightarrow =\int_0^\infty \frac{e^{bt}-e^{-bt}}{2}a^{-st}dt \Rightarrow$ by using (2) we get

$$L^*(\sinh bt)=\frac{b}{(s\ln a)^2-b^2}$$

8) $L^*(\cosh bt)=\int_0^\infty \cosh bt\, a^{-st}dt \Rightarrow =\int_0^\infty \frac{e^{bt}+e^{-bt}}{2}a^{-st}dt \Rightarrow$ by using (2) we get

$$L^*(\cosh bt)=\frac{s\ln a}{(s\ln a)^2-b^2}$$

5.How to Use (L.T) for Solving (L.O.D.E) Without Using any Initial Conditions

The Laplace transformations (L.T) are also used to solve (L.O.D.E)with constant coefficients and initial conditions. The last method is summarized by taking (L.T) of both sides of (L.O.D.E) and by substituting initial conditions and writing (L.T) for general solution and taking inverse Laplace transformation of both sides .So we can obtain the solution of (D.E) whose solution is required.

[Mohammed, A.H.] used Laplace transformations which have the form

$L\{f(x)\}=\int_0^\infty e^{-sx}f(x)dx$, for solving (L.O.D.Es) of order (n) ,its general form

$$a_1y^{(n)}+a_2y^{(n-1)}+...+a_ny=f(x)$$.

and by using (L.T) of both sides we can get

$$L(y)=\frac{K(s)}{(a_1s^n+a_2s^{n-1}+...+a_n)\bullet H(s)}$$

where $K(s)$ represents the result collections of numerator and denominator($L\cdot T$)to the function $f(x)$

and by taking (L^{-1}) of both sides of the above equation, we can obtain the following solution

$$y=A_1g_1(x)+A_2g_2(x)+...A_ng_n(x)+B_1h_1(x)+B_2h_2(x)+...+B_nh_n(x)$$

where $A_1, A_2, \cdots, A_n, B_1, B_2, \cdots, B_r$ are constants and , $g_1, g_2, \cdots, g_n, h_1, h_2, \cdots, h_r$ are functions of (x) .

6.Using New Transformation $(L^{*}T)$ For Solving (L.O.D.E)

suppose that general form of the linear ordinary differential equations of order (m) with constant coefficients and non-homogenous is:-

$$a_1 y^{(m)} + a_2 y^{(m-1)} + \dots + a_m y = g(x) \quad \dots (3$$

where $a_1, a_2, \cdots, a_m$ are constant coefficient.

Now by taking new transformation ($L^{*}T$) of both sides to the equation (3),and by using property(1) of lift sides to the equation (3), then we can get

$$L^{*}(y) = \frac{H_1(s)}{\left[a_1 (s \ln a)^m + a_2 (s \ln a)^{m-1} + \dots + a_m\right] \bullet H_2(s)} \quad \dots (4$$

where $H_1(s)$ represents the result collections of numerator and denominator($L^{*}T$)to the function $g(x)$ with $y(0), y'(0), \dots, y^{(m-1)}(0)$, and $H_2(s)$ represents denominator ($L^{*}T$)to the function $g(x)$.

Now,

$H_1(s)$, $H_2(s)$ are polynomial of (s) ,and [$a_1 (s \ln a)^m + a_2 (s \ln a)^{m-1} + \dots + a_m$], is also a polynomial of (s) and its degree (m) ,therefore its degree more than of $H_1(s)$,since our aim in this paper is to find the general solution of linear ordinary differential equation without using initial conditions therefore it is not necessary to know the terms of $H_1(s)$ .Here we only denoted to it by this symbol.

since (a) is a positive integer and let $(\ln a = d)$, hence the equation (4) becomes

$$L^{*}(y) = \frac{H_1(s)}{\left[a_1 (sd)^m + a_2 (sd)^{m-1} + \dots + a_m\right] \bullet H_2(s)} \quad \dots (5$$

Now by taking $(L^{*})^{-1}$ of both sides to equation (5), then we can get

$$y = (L^{*})^{-1} \left\{ \frac{H_1(s)}{\left[a_1 (sd)^m + a_2 (sd)^{m-1} + \dots + a_m\right] \bullet H_2(s)} \right\}$$

$$y = (L^{*})^{-1} \left\{ \frac{H_1(s)}{\left[(a_1 d^m) s^m + (a_2 d^{m-1}) s^{m-1} + \dots + a_m\right] \bullet H_2(s)} \right\}$$

let $a_1 d^m = c_1, a_2 d^{m-1} = c_2, \cdots, a_m d^{m-m} = c_m$, and by substituting in above equation, we get

$$y=\left(L^{*}\right)^{-1}\left\{\frac{H_{1}(x)}{\left[c_{1}s^{n}+c_{2}s^{n-1}+\dots+c_{n}\right]\cdot H_{2}(x)}\right\}$$

Now we can obtain the following solution:

$$y=A_{1}G_{1}(x)+A_{2}G_{2}(x)+\dots A_{n}G_{n}(x)+B_{1}H_{1}(x)+B_{2}H_{2}(x)+\dots+B_{n}H_{n}(x) \quad \dots (6$$

where $A_{1},A_{2},\dots A_{n},B_{1},B_{2},\dots B_{n}$ are constants and , $G_{1},G_{2},\dots G_{n},H_{1},H_{2},\dots H_{n}$ are functions of (x) .

Note that the order of equation (3) is (m) , therefore its general solution contains (m) constants. But the solution in (6) contains $(m+n+d)$ constants and to solve this problem we can eliminate some of these constants $B_{1},B_{2},\dots B_{n}$ and obtaining their values by substituting the solution (6) in equation $G(x,y',y'',\dots y^{(n)})=0$,so we can get a solution which contains (m) constants and it will be the required solution, and by using this method we can get the general solution of equation (3) without using any initial conditions but by using new transformation.

7.Examples:-

Example (7.1):-

To solve the (ODE) $y''+y'=\sin x$

By taking($L^{*}T$) of both sides

$$L^{*}(y)=\frac{H(s)}{\left[(sd)^{2}+1\right]\cdot\left[(sd)^{2}+sd\right]}$$

Now by taking $\left(L^{*}\right)^{-1}$ for the above equation

$$y=\left(L^{*}\right)^{-1}\left(\frac{H(s)}{\left[(sd)^{2}+1\right]\cdot\left[sd+1\right]\left[sd\right]}\right)$$

$$y=\left(L^{*}\right)^{-1}\left(\frac{A}{sd}\right)+\left(L^{*}\right)^{-1}\left(\frac{B}{sd+1}\right)+\left(L^{*}\right)^{-1}\left(\frac{Cx+D}{(sd)^{2}+1}\right)$$

$$y=A\left(L^{*}\right)^{-1}\left(\frac{1}{sd}\right)+B\left(L^{*}\right)^{-1}\left(\frac{1}{sd+1}\right)+\frac{c}{d}\left(L^{*}\right)^{-1}\left(\frac{sd}{(sd)^{2}+1}\right)+D\left(L^{*}\right)^{-1}\left(\frac{1}{(sd)^{2}+1}\right)$$

$$y=A+Be^{-x}+\frac{c}{d}\cos x+D\sin x$$

the above equation is contains (5) constants to solve this problem we can eliminate some of these constants

$$y'=-Be^{-x}-\frac{c}{d}\sin x+D\cos \quad , \quad y''=Be^{-x}-\frac{c}{d}\cos x-D\sin x$$

$$Be^{-x}-\frac{c}{d}\cos x-D\sin x-Be^{-x}-\frac{c}{d}\sin x+D\cos x=\sin x$$

$$\frac{-c}{d}+D=0,-D-\frac{c}{d}=1$$

by solving these equations we get $D=-\frac{1}{2},\frac{c}{d}=\frac{-1}{2}$

$$y=A+Be^{-x}-\frac{1}{2}\cos x-\frac{1}{2}\sin x$$

where A,B arbitrary constants.

Example (7.2):-

To solve the (ODE) $y'+y=e^{x}\cos x$

By taking($L^{*}T$) of both sides

$$L^{*}(y)=\frac{H_1(s)}{\left[(sd-1)^2+1\right]\bullet\left[sd+1\right]}$$

Now by taking $(L^{*})^{-1}$ for the above equation

$$y=(L^{*})^{-1}\left(\frac{H_1(s)}{\left[(sd-1)^2+1\right]\bullet\left[sd+1\right]}\right)$$

$$y=(L^{*})^{-1}\left(\frac{A}{sd+1}\right)+(L^{*})^{-1}\left(\frac{Bs+c}{(sd-1)^2+1}\right)$$

$$y=A(L^{*})^{-1}\left(\frac{1}{sd+1}\right)+e^{x}(L^{*})^{-1}\left(\frac{Bs+c}{(sd)^2+1}\right)$$

$$y=A(L^{*})^{-1}\left(\frac{1}{sd-1}\right)+\frac{B}{d}e^{x}(L^{*})^{-1}\left(\frac{sd}{(sd)^2+1}\right)+c\,e^{x}(L^{*})^{-1}\left(\frac{1}{(sd)^2+1}\right)$$

$$y=Ae^{x}+\frac{B}{d}e^{x}\cos x+c\,e^{x}\sin x$$

the above equation contains (4) constants and to solve this problem we can eliminate some of these constants

$$y'=Ae^{x}-\frac{B}{d}e^{x}\sin x+\frac{B}{d}e^{x}\cos x+ce^{x}\cos x+ce^{x}\sin x$$

$$Ae^{x}-\frac{B}{d}e^{x}\sin x+\frac{B}{d}e^{x}\cos x+ce^{x}\cos x+ce^{x}\sin x+Ae^{x}+\frac{B}{d}e^{x}\cos x+ce^{x}\sin x=e^{x}\cos x$$

$$-\frac{B}{d}+2c=0,\frac{2B}{d}+c=1$$

So, by solving these equations we can get $c = \frac{1}{5}, \frac{B}{d} = \frac{2}{5}$

$$y = Ae^{x} + \frac{2}{5}e^{x}\cos x + \frac{1}{5}e^{x}\sin x$$

where A is arbitrary constant.

Example (7.3):-

To solve the (ODE) $y''' - y'' = e^{-x}$

By taking($L^{*}T$) of both sides

$$L^{*}(y) = \frac{H_1(s)}{\left[(sd)^{3} - (sd)^{2}\right] \cdot [sd + 1]}$$

Now by taking $(L^{*})^{-1}$ for the above equation

$$y = (L^{*})^{-1}\left(\frac{H_1(s)}{[sd - 1] \cdot [sd + 1](sd)^{2}}\right)$$

$$y = (L^{*})^{-1}\left(\frac{A}{sd}\right) + (L^{*})^{-1}\left(\frac{B}{(sd)^{2}}\right) + (L^{*})^{-1}\left(\frac{C}{sd + 1}\right) + (L^{*})^{-1}\left(\frac{D}{sd - 1}\right)$$

$$y = A(L^{*})^{-1}\left(\frac{1}{sd}\right) + B(L^{*})^{-1}\left(\frac{1}{(sd)^{2}}\right) + C(L^{*})^{-1}\left(\frac{1}{sd + 1}\right) + D(L^{*})^{-1}\left(\frac{1}{sd - 1}\right)$$

$$y = A + BX + Ce^{-x} + De^{x}$$

the above equation contains (4) constants and solve this problem we can eliminate some of these constants

$$y' = B - Ce^{-x} + De^{x}, \quad y'' = Ce^{-x} + De^{x}, \quad y''' = -Ce^{-x} + De^{x}$$

$$-Ce^{-x} + De^{x} - Ce^{-x} - De^{x} = e^{-x}$$

So, by solving these equations we can get $-2C = 1 \Rightarrow C = \frac{-1}{2}$

$$y = A + BX + De^{x} - \frac{1}{2}e^{-x}$$

where A, B and D arbitrary constants.

Example (7.4):-

To solve the (ODE) $y'' + y' - 2y = 1$

By taking($L^{*}T$) of both sides

$$L^{*}(y) = \frac{H_1(s)}{\left[(sd)^{2} + sd - 2\right] \cdot sd}$$

Now by taking $(L^*)^{-1}$ for the above equation

$$y = (L^*)^{-1}\left(\frac{H_1(s)}{[sd-1][sd+2](sd)}\right)$$

$$y = (L^*)^{-1}\left(\frac{A}{sd}\right) + (L^*)^{-1}\left(\frac{B}{sd+2}\right) + (L^*)^{-1}\left(\frac{C}{sd-1}\right)$$

$$y = A(L^*)^{-1}\left(\frac{1}{sd}\right) + B(L^*)^{-1}\left(\frac{1}{sd+2}\right) + C(L^*)^{-1}\left(\frac{1}{sd-1}\right)$$

$$y = A + Be^{-2x} + Ce^{x}$$

the above equation contains (3) constants and to solve this problem we can eliminate some of these constants

$$y' = -2Be^{-2x} + Ce^{x}, \quad y'' = 4Be^{-2x} + Ce^{x}$$

$$4Be^{-2x} + Ce^{x} - 2Be^{-2x} + Ce^{x} - 2A - 2Be^{-2x} - 2Ce^{x} = 1$$

So, by solving these equations we can get $-2A = 1 \Rightarrow A = \frac{-1}{2}$

$$y = \frac{-1}{2} + Be^{-2x} + Ce^{x}$$

where B , C arbitrary constants.

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