Weyl' s theorem holds for algebraically *-paranormal operators

By
Shaima Showket Kadhim
University of Kufa –College of Engineering

المستخلص:

في هذا البحث نبرهن بان نظرية وايل متحققه للموثرات الجبريه شبه السويه ضمن شروط معينه. اذا كانت p(A) شبه سويه لبعض متعددات الحدود المعقده الغير ثابته p فان لكل $f \in H(\sigma(A))$ تكون نظرية وايل متحققه ا لـf(A) عندما تكون $H(\sigma(A))$ تمثل مجموعة الدوال التحليليه على الجوار المفتوح لمجموعة الطيف للموثر A وهي $\sigma(A)$.

Abstract : In this paper it is shown that if A is an "algebraically *-paranormal " operator , i.e., p(A) is *-paranormal for some nonconstant complex polynomial p, then for every $f \in H(\sigma(A))$, Weyl' s theorem holds for f(A), where $H(\sigma(A))$ denotes the set of analytic functions on an open neighborhood of $\sigma(A)$.

1. Introduction

Throughout this note let B(H) and K(H) denote , respectively , the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space H. If A $\in$ B(H) we shall write N(A) and R(A) for the null space and the range of A , respectively . Also , let $\alpha(A) = \dim N(A)$, $\beta(A) = \dim N(A^*)$, and let $\sigma(A), \sigma_a(A)$ and $\pi_o(A)$ denote the spectrum , approximate point spectrum and point spectrum of A , respectively .

For an operator A $\in$ B(H) , the ascent a(A) and the descent d(A) are given by

a(A) =inf $\{n \geq 0 : N(A^n) = N(A^{n+1})\}$ and d(A) = inf $\{n \geq 0 : R(A^n) = R(A^{n+1})\}$, respectively ; the infimum over the empty set is taken to be infinite . If the ascent and the descent of A $\in$ B(H) are both finite , then a(A) = d(A) =p , H = $N(A^p) \oplus R(A^p)$ and $R(A^p)$ is closed , [15] .

Also, an operator A $\in$ B(H) is called Fredholm if it has closed range , finite dimensional null space , and its range has finite co-dimension.

The index of a Fredholm operator is given by

i(A) = $\alpha(A) - \beta(A)$.

An operator A $\in$ B(H) is called Weyl if it is a Fredholm of index zero , and Browder if it is Fredholm "of finite ascent and descent" ; equivalently

[7] if A is Fredholm and $A-\lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in C$.

(2)

The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\omega(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by(cf. [6],[7])

$\sigma_e(A) = \{\lambda \in C : A-\lambda \text{ is not Fredholm}\}$,
$\omega(A) = \{\lambda \in C : A-\lambda \text{ is not Weyl}\}$,
$\sigma_b(A) = \{\lambda \in C : A-\lambda \text{ is not Browder}\}$,

respectively . Evidently

$\sigma_e(A) \subseteq \omega(A) \subseteq \sigma_b(A) = \sigma_e(A) \cup acc\,\sigma(A)$,

Where we write acc K for the accumulation points of K $\subseteq$ C . If we write iso K= K\acc K then we let

$\pi_{00}(A) = \{\lambda \in iso\,\sigma(A) : 0 < \alpha(A-\lambda) < \infty\}$,
and

$p_{00}(A) = \sigma(A) \setminus \sigma_b(A)$

We say that Weyl ' s theorem holds for A if

$\sigma(A) \setminus \omega(A) = \pi_{00}(A)$.

In this paper we investigate the validity of Weyl ' s theorem for algebraically *-paranormal operators.

We consider the sets

$\phi_+(H) = \{A \in B(H) : R(A) \text{ is closed and } \alpha(A) < \infty\}$,
$\phi_-(H) = \{A \in B(H) : R(A) \text{ is closed and } \beta(A) < \infty\}$,

and

$\phi_+^-(H) = \{A \in B(H) : A \in \phi_+(H) \text{ and } i(A) \le 0\}$.

By definition,

$\sigma_{ea}(A) = \cap\{\sigma_a(A+K) : K \in K(H)\}$

is the essential approximate point spectrum , and

$\sigma_{ab}(A) = \cap\{\sigma_a(A+K) : AK = KA \text{ and } K \in K(H)\}$

is the Browder essential approximate point spectrum.

In [12] , it was shown that $\sigma_{ea}(A) = \{\lambda \in C : A - \lambda \notin \Phi_+^-(H)\}$

In [16], Weyl proved that Weyl ` s theorem holds for hermitian operators. Weyl ` s theorem has been extended from hermitian operators to hyponormal and Toeplitz

(3)

operators [3] , and to several classes of operators including semi-normal operators ([1],[2]) . Recently , the second named author W.Y.Lee [5] showed that Weyl ` s theorem holds for algebraically hyponormal operators. In this paper , we extend this result to algebraically *-paranormal operators.

2- Preliminaries

Definition 1: An operator A is said to be *-paranormal if $\|A^*x\|^2 \le \|A^2x\|\|x\|$ for all $x \in H$.

Proposition 1 [14] : If A is *-paranormal , then $\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\}$.

3- Main results

We say that A is algebraically *-paranormal if there exists a nonconstant complex polynomial p such that p(A) is *-paranormal.
The following implications hold:
hyponormal $\Rightarrow$ *-paranormal $\Rightarrow$ algebraically *-paranormal.

Lemma 3.1: If A is invertible and *-paranormal then A^{-1} is *-paranormal.

Proof: Given $x \in H$ let $y = A^{-1}x$ and $z = A^{-1}y$, so $Az = y$ and $A^2z = x$. Then

$$\|A^{-1*}x\|^2 = \|(A^{-1}x)^*\|^2 = \|(y)^*\|^2 = \|(Az)^*\|^2 = \|A^*z\|^2 \le \|A^2z\|\|z\|$$

$$= \|x\|\|A^{-2}x\| = \|(A^{-1})^2x\|\|x\|.$$

Lemma 3.2 : Let $A \in B(H)$ be a *-paranormal operator and $M \subset H$ be an invariant subspace of A. Then the restriction $A|M$ to its invariant subspace M is also *-paranormal .

Proof : Let $x \in M$ be an arbitrary vector . Then we have ,

$$\|(A|M)^* x\|^2 = \|A^*|M\, x\|^2 = \|A^* x\|^2 \le \|A^2 x\|\,\|x\| = \|(A|M)^2 x\|\,\|x\|$$

This implies that $A|M$ is *-paranormal.

Definition 2 : An operator A is called isoloid if every isolated point of $\sigma(A)$ is an eigenvalue of A.

Theorem 1 : If A is *-paranormal, then A is isoloid.

Proof: Let $\lambda \in \sigma(A)$ be an isolated point , then the range of Riesz projection $E = \frac{1}{2\pi i}\int_{\partial D}(z-A)^{-1}dz$ is an invariant closed subspace of A and $\sigma(A|EH) = \{\lambda\}$, where D

(4)

is a closed disk with its center λ such that $\sigma(A) \cap D = \{\lambda\}$. If $\lambda = 0$, then $\sigma(A|EH) = \{0\}$.

Since $A|EH$ is *-paranormal by Lemma 3.2, $A|EH = 0$ by proposition 1. Therefore 0 is an eigenvalue of A . If $\lambda \neq 0$, then $A|EH$ is an invertible *-paranormal operator and hence $(A|EH)^{-1}$ is also *-paranormal by Lemma 3.1. By proposition 1 , we see $\|A|EH\| = |\lambda|$ and $\|(A|EH)^{-1}\| = \frac{1}{|\lambda|}$ .Let $x \in EH$ be an arbitrary vector . Then $\|x\| \le \|(A|EH)^{-1}\|\,\|A|EH\,x\| = \frac{1}{|\lambda|}\|A|EH\,x\| \le \frac{1}{|\lambda|}|\lambda|\|x\| = \|x\|$. This implies that $\frac{1}{\lambda}A|EH$ is unitary with its spectrum $\sigma\left(\frac{1}{\lambda}A|EH\right) = \{1\}$. Hence $A|EH = \lambda$ and λ is an eigenvalue of A . This completes the proof .

We write r(A) and W(A) for the spectral and numerical range of A , respectively . It is well known that r(A) $\le \|A\|$ and that W(A) is convex with convex hull conv $\sigma(A) \subseteq \overline{W(A)}$. A is called convexoid if conv $\sigma(A) = \overline{W(A)}$ and normaliod if r(A) = $\|A\|$.

Lemma 3.3: Let A be a*- paranormal operator, $\lambda \in C$,and assume that $\sigma(A)=\{\lambda\}$. Then $A=\lambda$.

Proof : By following the same way in [13, lemma 2.1] .

In [4] , B.P.Duggal and S.V. Djordjevic proved that quasinilpotent algebraically *-paranormal operators are nilpotent .We now establish a similar result for algebraically *-paranormal operators.

Lemma 3.4: Let A be aquasinilpotent algebraically *-paranormal operator. Then A is nilpotent.

Proof : By following the same way in [13, lemm 2.2] .

We say that $A \in B(H)$ has the single valued extension property (SVEP) if for every open set $U \subseteq C$ the only analytic function $f: U \to H$ which satisfies the equation $(A-\lambda)f(\lambda)=0$ is the constant function $f=0$.

Lemma 3.5: Let $A \in B(H)$ be an algebraically *-paranormal operator . Then A has finite ascent . In particular, every algebraically *-paranormal operator has SVEP.

Proof : Suppose p(A) is *-paranormal for some nonconstant polynomial p. Since *-paranormal is translation-invariant, we may assume p(0)=0. If $p(\lambda)=a_0\lambda^m$, then

(5)

$\ker(A^m)=\ker(A^{2m})$ because *-paranormal operators are of ascent 1 . Thus we write $p(\lambda)=a_0\lambda^m(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)(m\neq 0; \lambda_i\neq 0 \ for\ 1\leq i\leq n)$. We then claim that

$\ker(A^m)=\ker(A^{m+1})$ (3.1)

To show (3.1) , let $0\neq x\in \ker(A^{m+1})$. Then we can write

$p(A)x=(-1)^n a_0\lambda_1\cdots\lambda_n A^m x.$

Thus we have

$$\left|\alpha_0 \lambda_1 \cdots \lambda_n\right|^2 \left\|A^m x\right\|^2 = \left(p(A)x, p(A)x\right)$$

$$\leq \left\|p(A)^* p(A)x\right\| \|x\|$$

$$\leq \left\|p(A)^* p(A)x\right\|^2 \|x\|$$

$$\leq \left\|p(A)^2\right\| \|x\| \; (because\ p(A)\ is\ * - paranormal)$$

$$= \left\|\alpha_0{}^2 (A - \lambda_1 I)^2 \cdots (A - \lambda_n I)^2 A^{2m} x\right\| \|x\|$$

$$= 0,$$

which implies $x \in \ker\left(A^m\right)$ Therefore $\ker\left(A^{m+1}\right) \subseteq \ker\left(A^m\right)$ and the reverse inclusion is always true . Since every algebraically *-paranormal operator has finite ascent , it follows from [9] that every algebraically *-paranormal operator has SVEP.

From the Theorem 1 ,we obtain that every *-paranormal operator is isoloid .We now extend this result to algebraically *-paranormal operators.

Theorem 2: Let A be an algebraically *-paranormal operator . Then A is isoloid.

Proof : Let $\lambda \in iso\,\sigma(A)$ and let $E = \frac{1}{2\pi i}\int_{\partial D}(z - A)^{-1} dz$ be the associated Riesz idempotent , where D is a closed disk centered at λ which contains no other points of $\sigma(A)$. We can then represent A as the direct sum $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where $\sigma(A_1) = \{\lambda\}$ and $\sigma(A_2) = \sigma(A) \setminus \{\lambda\}$.
Since A is algebraically *-paranormal , p(A) is *-paranormal for some nonconstant polynomial p. Since $\sigma(A_1) = \{\lambda\},$ we must have $\sigma(p(A_1)) = p(\sigma(A_1)) = \{p(\lambda)\}.$ Therefore $p(A_1) - p(\lambda)$ is quasinilpotent. Since $p(A_1)$ is *-paranormal , it follows from Lemma 3.3 that $p(A_1) - p(\lambda) = 0.$ Put $q(z) = p(z) - p(\lambda).$ Then $q(A_1) = 0,$ and hence A_1 is algebraically *-paranormal . Since $A_1 - \lambda$ is quasinilpotent and algebraically *-paranormal , it follows from Lemma 3.4 that $A_1 - \lambda$ is nilpotent . Therefore $\lambda \in \pi_0(A_1),$ and hence $\lambda \in \pi_0(A).$ This shows that A is isoloid.

Theorem 3: Weyl´s theorem holds for every algebraically *-paranormal operator.

(6)

Proof: Suppose p(A) is *-paranormal for some nonconstant polynomial p. We first prove that $\pi_{00}(A)\subseteq\sigma(A)\setminus\omega(A)$. Since algebraically *-paranormal is translation-invariant, it suffices to show that $0\in\pi_{00}(A)\Rightarrow A$ is Weyl but not invertible. Suppose $0\in\pi_{00}(A)$. Now using the spectral projection $E=\frac{1}{2\pi_i}\int_{\partial D}(z-A)^{-1}dz$, where D is a closed disk centered at 0 which contains no other points of $\sigma(A)$. As before , we can represent A as the direct sum $A=\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}$, where $\sigma(A_1)=\{0\}\ and\ \sigma(A_2)=\sigma(A)\setminus\{0\}$.

But then A_1 is also algebraically *-paranormal and quasinilpotent. Thus by Lemma 3.4, A_1 is nilpotent . Thus we should have that $\dim R(P)<\infty$; if it were not so, then $N(A_1)$ would be infinite dimensional operator . Since finite dimensional operators are always Weyl it follows that A_1 is Weyl. But since A_2 is invertible we can conclude that A is Weyl . Therefore $\pi_{00}(A)\subseteq\sigma(A)\setminus\omega(A)$. For the reverse inclusion, suppose $\lambda\in\sigma(A)\setminus\omega(A)$. Thus $A-\lambda I$ is Weyl . Then by the "Index Product Theorem",

$$\dim N\left((A-\lambda I)^n\right)-\dim R\left((A-\lambda I)^n\right)^{\perp}=i\left((A-\lambda I)^n\right)=n\,i(A-\lambda I)=0.$$

Thus if $\dim N\left((A-\lambda I)^n\right)$ is a constant , then so is $\dim R\left((A-\lambda I)^n\right)^{\perp}$. Consequently finite ascent forces finite descent. Therefore by Lemma 3.5 , $A-\lambda I$ is Weyl of finite ascent and descent , and thus it is Browder. Therefore $\lambda\in\pi_{00}(A)$. This completes the proof.

Theorem 4: If A is an algebraically *-paranormal operator , then $\omega(f(A))=f(\omega(A))$ for every $f\in H(\sigma(A))$, where $H(\sigma(A))$ is the space of functions analytic in an open neighborhood of $\sigma(A)$.

Proof: Since $\omega(f(A))\subseteq f(\omega(A))$ with no other restriction on A , it suffices to show that $f(\omega(A))\subseteq\omega(f(A))$. A necessary and sufficient condition for equality in the above inclusion is that $i(A-\lambda I).\,i(A-\mu I)\geq 0$ for each pair of complex numbers λ,μ which are not in $\sigma_e(A)$ (see[8]). Let A be an algebraically *-paranormal operator ; then by Lemma , A- λI has finite ascent for every λ , and so if $A-\lambda I$ is Fredholm then $i(A-\lambda I)\leq 0$. Now, if $A-\lambda I$ has finite descent , then $i(A-\lambda I)=0$, and if $A-\lambda I$ does not have finite descent, then

$i(A-\lambda I) < 0 \left(\begin{array}{l} \text{since } i(A-\lambda I) = \dim N(A-\lambda I)^n - \text{codim } R(A-\lambda I)^n \to -\infty \\ , \text{as } n \to \infty \end{array} \right)$

This completes the proof.

Corollary 4.1: If A is an algebraically *-paranormal operator , then for every $f \in H(\sigma(A))$,Weyl΄ s theorem holds for $f(A)$.

(7)

Proof: Remembering ([11]) that if A is isoloid , then $f(\sigma(A) \setminus \pi_{00}(A)) = \sigma(f(A)) \setminus \pi_{00}(f(A))$ for every $f \in H(\sigma(A))$;
it follows from Theorem 1 , Theorem 3 and Theorem 4 that $\sigma(f(A)) \setminus \pi_{00}(f(A)) = f(\sigma(A) \setminus \pi_{00}(A)) = f(\omega(A)) = \omega(f(A))$,
which implies that Weyl΄ s theorem holds for $f(A)$.

Theorem 5: If A $\in B(H)$ is algebraically *-paranormal ,then $\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$ for every $f \in H(\sigma(A))$.

Proof: Note that it is enough to prove the inclusion $f(\sigma_{ea}(A)) \subset \sigma_{ea}(f(A))$. Suppose that $\lambda \notin \sigma_{ea}(f(A))$. Then $f(A) - \lambda \in \phi_+^-(H)$ and $f(A) - \lambda = a_0(A-\lambda_1) \cdots (A-\lambda_k)$, where $a_0 \in C$ and $A - \lambda_i \in \phi_+(H)$ .Arguing as in the proof of Theorem 4 , we have that $i(A-\lambda_i) \leq 0$ and hence that $A - \lambda_i \in \phi_+^-(H)$ for all i =1,2,…,k. This implies that $\lambda \notin f(\sigma_{ea}(A))$.

4- References

1. S.K.Berberian, An extension of Weyl΄ s theorem to a class of not necessarily normal operators, Michigan Math.J. 16 (1969) , 273-279.
2. S.K.Berberian, The Weyl spectrum of an operator , Indiana Univ. Math. J . 20 (1970) , 529-544.
3. L.A.Coburn,Weyl΄ s theorem for nonnormal operators , Michigan Math . J. 13 (1966) , 285-288.
4. B.P.Duggal and S.V.Djordjevic,Weyl΄ s theorem in the class of algebraically p-hyponormal operators, Comment.Math.PraceMat. 40 (2000),49-56.
5. Y.M.Han and W.Y.Lee, Weyl΄ s theorem holds for algebraically hyponormal operators , Proc. Amer.Math.Soc.128(2000),2291-2296.
6. R.E.Harte,Fredholm,Weyl and Browder theory , Proc. Royal Irish Acad.85A (1985), 151-176.

7. R.E.Harte, Invertibility and Singularity for bounded Linear Operators , Dekker, New York, 1988.

8. R.E.Harte and W.Y.Lee, Another note on Weyl' s theorem,Trans.Amer.Math. Soc. 349 (1997), 2115-2124.
9. K.B.Laursen, Operators with finite ascent, Pacific J . Math. 152 (1992) , 323-336.
10. W.Mlak, Hyponormal contractions , Colloq.Math.18 (1967) , 137-141.
11. K.K.Oberai, On the Weyl spectrum (II), Illinois J. Math.21 (1977) , 84-90 . MR 55:1102
12. V.Rakocevic , On the essential approximate point specxtrum II , Mat. Vesnik.36 (1984) , 89-97.

13. Raul E. Curto and Young Min Han, Weyl' s theorem for algebraically paranormal operators, Integr. equ . oper.theory 47 (2003) 307-317.
14. C.S.Ryoo, Some class of operators, Math . J.Toyama Univ. 21 (1998) , 147-152.
15. Taylor, A., Lay, D., Introduction to functional analysis , 2nd ed ., John Wiley & Sons , New York – Chichester-Brisbane, 1980.

16. H.Weyl, Ü ber beschrankte quadratische Formen , deren Differenz vollsteig ist,Rend. Circ.Mat.Palermo 27 (1909) , 373-392.