Sum and Direct Sum in VN-Regular Fuzzy Ring

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Abstract:-

In this paper, by using defintions of fuzzy ring , fuzzy ideals , Von Neumann regular fuzzy ring and some properties about them we study sum and direct sum of fuzzy ring and VN-regular fuzzy ring .

1.Introduction:-

The concept of fuzzy set defined by Zadeh,[1] ,he define fuzzy singleton and fuzzy zero singleton of a fuzzy number .The fuzzy ring and fuzzy ideal were introduced by [Liu,1982] .Also, we define regular fuzzy ring and Von Neumann regular fuzzy ring,which denoted by VN-regular fuzzy ring .We will define the sum and direct sum of fuzzy ideals of fuzzy ring of a ring R and prove that :-A=J+K if and only if $A_t=(J+K)_t= J_t+K_t$, and $J \cap K =0_t$ if and only if $(J \cap K)_t =0= J_t \cap K_t$. A=J+K and $J \cap K=0_t$ if and only if $A_t= J_t+K_t$ and $J_t \cap K_t =0$. $A=J \oplus K$ if and only if $A_t =(J \oplus K)_t = J_t \oplus K_t$ for all $t\in[0,1]$,when A_t is the level of A also, J_t , K_t are the levels of J,K respectively.Beside that we will prove some properties about sun and direct sum of VN-regular fuzzy ring .

2.Fzzy Set and Fuzzy Ring .

Definition 2.1 [1] :-Let R be a non – empty set , I be the closed interval [0,1] of the real line (real number) . **A fuzzy set** A in R (a fuzzy subset A of R) is characterized by a membership function A:R →I which is associated with each point $x \in R$ its grade or degree of membership
$A(x) \in [0,1]$.

Definition 2.2[1]:-Let x_t: R → I be a fuzzy set in R , where $x\in R$, for all $t \in [0,1]$, define by

$$x_t(y) = \begin{cases} t & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}$$

for all $y \in R$. Than x_t is called a **fuzzy singleton** .
Let R = 1R and if x = 0 and t =1 , then

$$0(y) = \begin{cases} 1 & \text{if } y=0 \\ 0 & \text{if } y\neq 0 \end{cases}$$

We shall call such fuzzy singleton . **The fuzzy zero singleton** of fuzzy numbers .

Definition 2.3 [2] :- Let (R,+,.) be a ring and let A be a fuzzy set in R . Then A is called **fuzzy ring** in a ring (R,+,.) , if and only if for each
$x ,y \in R$:-

1- $A(x+y) \geq \min\{A(x), A(y)\}$.
2- $A(x) = A(-x)$.
3- $A(xy) \geq \min\{A(x), A(y)\}$.

Definition 2.4 [5] :- Let A be a fuzzy ring . If $t \leq A(0)$ then the set $\{x \in R : A(x) \geq t\}$ is a subring of R which is called the level subring and denoted by A_t .

Definition 2.5 [2] :-A fuzzy set A of a ring R is called a fuzzy ideal of R if for each x ,y ∈ R :-

1-$A(x-y) \geq \min\{A(x), A(y)\}$.
2- $A(xy) \geq \max\{A(x), A(y)\}$.

Definition 2.6 [4] :- Let A be a fuzzy left (right) ideal of R ,then the left (right) ideals A_t, where $t \in [0,1]$ are called level ideals.

Theorem 2.7[3]: Let $A : R \to I$ be a fuzzy ring and J,K be two fuzzy ideals of A.Then ,

1- $J \cap K$ is a fuzzy ideal of A .
2- $J \cup K$ is a fuzzy ideal of A if $J \subseteq K$ or $K \subseteq J$.

3. Sum and Direct Sum in VN-Regular Fuzzy Ring

In this section we define sum and direct sum in VN-regular fuzzy ring and study some properties about them .

Definition 3.1[3]:- Let J,K be two fuzzy ideals of a fuzzy ring A of a ring R.Then the sum two ideals J+K is define as :-

$$(J+K)(x)=\sup\{\min\{J(a),K(b)\} \mid x=a+b, \ \forall a,b \in R\}.$$

Proposition 3.2[3]:- Let J,K be two fuzzy ideals of a fuzzy ring A of a ring R.Then J+K is fuzzy ideal of A .

Definition 3.3 [4] :-Let A be a fuzzy ring of a ring R and $x_t \in A$ with $t \in [0,1]$, x_t is said to be **a regular fuzzy element (singleton)** if there exists $y_t \in A$, with $s \in [0,1]$, such that $x_t = x_t . y_s . x_t$

Definitions 3.4[5] :-A is called **VN-regular fuzzy ring** if and only if every fuzzy singleton in A is a regular fuzzy element (singleton) ,
i.e $\forall x_t \in A$, with $t \in [0,1]$, $\exists y_s \in A$, with $s \in [0,1]$, such that $x_t = x_t . y_s . x_t$

Proposition 3.5**:-** A is VN-regular fuzzy ring of R if and only if A_t is VN regular subring of R,for each $t \in [0,1]$.

Definitions 3.6 [5]:- Let A be fuzzy ring of R , $x_t \subseteq A$ where $t \in [0,1]$. x_t is said to be idempotent fuzzy singleton if and only if $x_t=(x_t)^2$ where $(x_t)^2 = x_t . x_t = (x\,x)_t = x_t$.

Definition 3.7[5]:-Given an idempotent fuzzy singleton e_t ,we define $A.e_t$ to be the set of all x_t in A with $e_t . x_t = x_t. e_t = x_t$.(i.e $A.e_t = \{ x_t \in A: x_t = e_t . x_t = x_t. e_t \}$) .

Proposition 3.8[5]:-Let A be a VN-regular fuzzy ring of a ring R,then $A.e_t$ is also VN-regular fuzzy ring.

Proposition 3.9:- Level of the coset is coset of the level. i.e $(a_tA)_t = aA_t$ for each $t \in [0,1]$.

Proof:- See [**5**]

Preposition 3.10:- In VN-regular fuzzy ring every finitely generated right(left) fuzzy ideal is principal.

Proof: Let A be a VN-regular fuzzy ring and let the left fuzzy ideal a_tA+b_tA implies that A_t is VN-regular subring and $aA_t +bA_t$ is teft ideal in A_t, then $aA_t+bA_t=eA_t$ implies that $(a_tA+b_tA)_t =(e_tA)_t$ for all $t \in [0,1]$. Since $(a_tA+b_tA)_t= aA_t +bA_t$ and $(e_tA)_t=eA_t$ .So $a_tA+b_tA = e_tA$ for all $t \in [0,l]$.

Definition 3.11:-Let A be a fuzzy ring of a ring R. A is called fuzzy ring dirict sum of two fuzzy ideals $J,K \subseteq A$ such that $A=J \oplus K$ if and only if $A=J+K$ and $J \cap K=0_t$ **.**

Proposition 3.12[5]:-If $J,K \subseteq A$ are two fuzzy ideals in A and $A=J \oplus K$ then $A_t=(J \oplus K)_t$.and

1.$(J+K)_t =J_t+K_t$ 2. $J \cap K =0_t$ if and only if $J_t \cap K_t =0$

Proposition 3.13:- A fuzzy ring A of a ring R isVN-regular fuzzy ring if and only if every principal fuzzy ideal is a direct sum.

Proof:Let A be a VN-regular fuzzy ring. Then A_t isVN-regular subring of a ring R implies that $eA_t=aA_t$ for any $a \neq 0$ in $A_t \subseteq R$ Hence A_t is a direct sum, i.e $A_t = aA_t \oplus (1 -e)A_t$ .Since $aA_t=(a_t A)_t$ and $(1-e)A_t=((1 -e)_t A)_t$ .So $A= a_t A \oplus (1-e)_t A$ for all $t \in [0,1]$. So $a_t A$ is a direct sum.

Concersely, let $A= a_t A \oplus J$ for every principale fuzzy ideal J of A. Then $A_t = aA_t \oplus J_t \subseteq R$ for all $t \in [0,1]$, then $l=ar \oplus k$ for some $a,r \in A_t$ and $k \in J_t$,implies that $a=ara \oplus ka$. Since $ka \in aA_t \cap J_t =0$,then a=ara implies that A_t is VN-regular subring. Then A is VN-regular fuzzy ring .

Proposition 3.14:- A finite direct sum of VN-regular fuzzy rings is VN-regular fuzzy ring .

Proof:- By definition $A_i : R_i \quad I$ (i=1,2,.....,n).$(\oplus A_i)(\oplus(x_i)) = \oplus A_i(x_i)$, i=1,2,...,n . Then it is clear that :-
1. $(\oplus A_i)(\oplus(x_i - y_i)) \geq \min\{(\oplus A_i)(\oplus x_i), (\oplus A_i)(\oplus y_i)\}$. 2. $(\oplus A_i)(\oplus(x_i . y_i)) \geq \min\{(\oplus A_i)(\oplus x_i), (\oplus A_i)(\oplus y_i)\}$.
Therefore $\oplus A_i$ is a fuzzy ring .
To prove that $\oplus A_i$ is VN-regular fuzzy ring ,when each A_i is VN-regular fuzzy ring (i=1,2,...,n).Let $\oplus a_i \in \oplus A_i$. Then for each $a_i \in A_i$ there exists $b_i \in A_i$ such that $a_i = a_i.b_i.a_i$ as a fuzzy singleton.Therefore $\oplus b_i \in \oplus A_i$ such that $\oplus a_i = \oplus(a_i.b_i.a_i) = (\oplus a_i)(\oplus b_i)(\oplus a_i) \in \oplus A_i$. This implies that $\oplus A_i$ is VN-regular fuzzy ring .

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