Some Properties on Jacobson Radical VN-Regular Fuzzy Ring

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Abstract:-

In this paper we will define Von Neumann regular fuzzy ring ,maximal fuzzy ideal and Jocobson radical denoted by J(A)and prove if A is a VN- regular fuzzy ring , then A has zero Jocobson radical,(i.e J(A)= 0_t) , for all $t \in [0,1]$ and relution between maximal fuzzy ideal ,prime fuzzy ideal and Jocobson radical .

1.Introduction :-

In [Zadeh,1965] ,Zadeh introduced the fundamental concept of fuzzy set,he define fuzzy singleton and fuzzy zero singleton of a fuzzy number.The fuzzy ring and fuzzy ideal were introduced by Liu in [Liu,1982] .Also, we define regular fuzzy ring and Von Neumann regular fuzzy ring,we denoted by VN-regular fuzzy ring we get the result: every fuzzy ring is VN–regular fuzzy ring of a ring if and only every level subring is VN-regular subring .In this paper we will definen Jocobson radical denoted by J(A)and prove if A is a VN- regular fuzzy ring , then A has zero Jocobson radical,(i.e J(A)= 0_t),for all $t\in[0,1]$and other result .

Definition1.1 [1] :-Let R be a non – empty set , I be the closed interval [0,1] of the real line (real number) . **A fuzzy set** A in R (a fuzzy subset A of R) is characterized by a membership function $A:R \to I$ which is associated with each point $x \in R$ its grade or degree of membership
$A(x) \in [0,1]$.

Definition 1.2[1]:-Let x_t: $R \to I$ be a fuzzy set in R , where $x\in R$, for all $t \in [0,1]$, define by

$$x_t(y) = \begin{cases} t & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}$$

for all $y \in R$. Than x_t is called a **fuzzy singleton** .
Let R = 1R and if x = 0 and t =1 , then

$$0_t(y) = \begin{cases} 1 & \text{if } y=0 \\ 0 & \text{if } y\neq 0 \end{cases}$$

We shall call such fuzzy singleton . **The fuzzy zero singleton** of fuzzy numbers .

2.Fuzzy Ring and VN-Regular Fuzzy Ring .

We will define fuzzy ring and VN-regular fuzzy ring and some properties about them .

Definition 2.1 [2] :-Let (R,+,.) be a ring and let A be a fuzzy set in R . Then A is called **fuzzy ring** in a ring (R,+,.),if and only if for each $x ,y \in R$:-

1- $A(x+y) \geq \min\{A(x), A(y)\}$.
2- $A(x) = A(-x)$.
3- $A(xy) \geq \min\{A(x), A(y)\}$.

Proposition 2.2 [2] :- A fuzzy set $A : R \rightarrow I$ with $A \neq 0$ is a fuzzy ring if and only if A_t is a subring of R, for all $t \in [0,A(0)]$ which is called the level subring .

Definition 2.3 [2] :-A fuzzy set A of a ring R is called a fuzzy ideal of R if for each x ,y ∈ R :-

1-$A(x-y) \geq \min \{ A(x) , A(y) \}$.

2- $A(xy) \geq \max \{ A(x) , A(y) \}$.

Definition 2.4 [4] :- Let A be a fuzzy left (right) ideal of R .Then the left (right) ideals A_t, where $t \in [0,1]$ are called level ideals.

Definition 2.5 [4] :-Let A be a fuzzy ring of a ring R and $x_t \in A$ with $t \in [0,1]$, x_t is said to be **a regular fuzzy element (singleton)** if there exists $y_t \in A$, with $s \in [0,1]$, such that $x_t = x_t . y_s . x_t$

Definitions 2.6[5]:-Let A is called **VN-regular fuzzy ring** if and only if every fuzzy singleton in A is a regular fuzzy element (singleton) ,
i.e $\forall$ x∈A, with $t \in [0,1]$, $\exists$ $y_s \in A$, with $s \in [0,1]$, such that $x_t = x_t . y_s . x_t$

Proposition 2.7[5]:- A is VN-regular fuzzy ring of R if and only if A_t is VN regular subring of R,for each $t \in [0,1]$.

Proposition 2.8:- Level of the coset is coset of the level. i.e $(a_t A)_t = aA_t$ for each $t \in [0,1]$.
Proof:- See[5].

Proposition 2.9[5]:-Let A be a VN-regular fuzzy ring of a ring R,then also $A.e_t$ is VN-regular fuzzy ring . Where $A.e_t = \{ x_t \in A: x_t = e_t \ x_t = x_t . e_t \}$.

3.Fuzzy Maximal Ideal and Jacobson Radical of Fuzzy Ring .

In this section We will define fuzzy maximal ideal and Jacobson radical of fuzzy ring and we prove some properties about them .

Definition 3.1[3] :- Let A be a fuzzy ideal of a ring (R,+,.) .Then A is called a fuzzy maximal ideal of R if A is not constant and for any fuzzy ideal B of R,if $A \subseteq B$ then either $A_t = B_t$ or $B = \lambda_R$.(where λ_R denotes the characteristic function of R such that :-

$$\lambda_R(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{if } x \notin R \end{cases}$$

Definition 3.2[2] :-Let A be a fuzzy ideal of R . Then A is called prime fuzzy ideal if either $A = \lambda_R$ or:-

1- A is not constant and

2-for any fuzzy ideals B,C of R .If $B \circ C \subseteq A$ then either $B \subseteq A$ or $C \subseteq A$.

Proposition 3.3[3]:- Let A be a fuzzy maximal ideal of a ring with identity (R,+,.).Then A is fuzzy prime ideal of R .

Proof :-Since A is a fuzzy maximal ideal and A(0)=1 ,
So A_t is maximal ideal of R.Since $1 \in R$,then A_t is prime ideal of R.
Hence A is fuzzy prime ideal of R.

Definition 3.4:- The Jacobson radical of a fuzzy ring A of a ring R , denoted by J(A), is the intersection of all maximal fuzzy ideals J of A .

Proposition 3.5:- If A is a VN- regular fuzzy ring , then A has zero Jocobson radical ,(i.e $J(A)= 0_t$) , for all $t \in [0,1]$.

Proof : Let x_t be an element of J(A) , for all $t \in [0,1]$.Since A is VN–regular fuzzy ring ,then A_t is VN–regular subring .Since $x_t \subseteq A$ so $x \in A_t$, $\exists\, y \in A_t$ such that $x=xyx \in A_t$ .Let e=yx be an idempotent element in A_t . Now , for any $x \in A_t$, there exists $z \in A_t$ such that e z - z = x . Consequently , yx=y e z – yz=0. Thus $x A_t = 0$ then $x_t A = 0_t$ implying that $x_t = 0_t$.

Remark :-The Jacobson radical J of a fuzzy ring A of a ring (R,+,.)is fuzzy ideal .

Proposition 3.6[3]:- Let (R,+,.) be a ring with identity , and let $\Im$ denote the family of all fuzzy maximal ideals of R.Then $J=(\cap\{M : M \in \Im\})_t$.

Proof :-Since $(\cap\{M : M \in \Im\})_t = \cap\{M_t : M \in \Im\}$ .If I is a maximal ideal of R,then $\lambda_I \in \Im$ and $(\lambda_I)_t = I$. Also ,if $M \in \Im$,then M_t is maximal ideal of R. Hence $J=(\cap\{M : M \in \Im\})_t = \cap\{M_t : M \in \Im\}$.

Proposition 3.7:- Let I and K be fuzzy ideals of A such that I(0)= 1=K(0) . Then $J(I \cap K)=J(I) \cap J(K)$
Proof :- See [3] .

Remarks :- 1- If A is a constant fuzzy ideal of R . Then $J(A)= \lambda_R$.
Proof:- Since **A** be constant then $A_t = R$ and $A= \lambda_R$ **.** Hence $J(A)= \lambda_R$ **.**

2- If A is a prime fuzzy ideal of a ring R . Then
J(A)= J(J(A)).
Proof:- Similarly as prove remark (1) .

-:References

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